

CONSTRUCTING THE HYPERBOLIC PLANE AS THE REDUCTION OF A THREE-BODY PROBLEM

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ABSTRACT. We construct the hyperbolic plane with its geodesic flow as the scale plus symmetry reduction of a three-body problem in the Euclidean plane. The potential is $-I/\Delta^2$ where I is the triangle's moment of inertia and Δ its area. The reduction method uses the Jacobi-Maupertuis metric, following [5].

1. THE PROBLEM.

Three point particles move in the Euclidean plane $\mathbb{R}^2 = \mathbb{C}$ according to Newton's equations

$$(1) \quad m_a \ddot{q}_a = -\nabla_a V, a = 1, 2, 3,$$

with potential

$$(2) \quad V(q) = -\frac{\gamma \text{ (moment of inertia)}}{(\text{area})^2} = -\frac{\gamma I(q)}{\Delta(q)^2}.$$

The m_a are the masses. The $q_a \in \mathbb{R}^2 = \mathbb{C}$ are the instantaneous positions of these point masses. The $\nabla_a = \frac{\partial}{\partial q_a}$ are the gradients with respect to q_a . We write $q = (q_1, q_2, q_3) \in \mathbb{C}^3$ for the three positions put together into one vector. The denominator of our potential is

$$\Delta(q) = \text{signed area of triangle of triangle with vertices } q_1, q_2, q_3,$$

while its numerator is

$$I(q) = \text{moment of inertia} = \frac{\sum_{a < b} m_a m_b r_{ab}^2}{\sum_a m_a}, \text{ where } r_{ab} = |q_a - q_b|,$$

the moment of inertia of this triangle with respect to its center of mass. The constant $\gamma > 0$ is a physical constant needed to make the units of the potential that of energy, so that γ has units of $(\text{length})^4/(\text{time})^2$.

Our ODEs form a Galilean-invariant Hamiltonian system, and as such have the usual conserved quantities

$$H = K(v) + V(q), P = \sum m_a v_a, J = \sum m_a q_a \wedge v_a,$$

of energy, linear momentum and angular momentum. In the expression for energy the term K is the kinetic energy

$$K(v) = \frac{1}{2} \sum m_a |v_a|^2 = \frac{1}{2} \langle v, v \rangle,$$

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a function of the velocities $v = \dot{q} \in \mathbb{C}^3$. The inner product occurring here is the “mass inner product” on $\mathbb{C}^3 = (\mathbb{R}^2)^3$,

$$\langle q, v \rangle = m_1 q_1 \cdot v_1 + m_2 q_2 \cdot v_2 + m_3 q_3 \cdot v_3.$$

where the dot product $q_i \cdot v_i = \operatorname{Re}(q_i \bar{v}_i)$ is the usual dot product in $\mathbb{R}^2 \cong \mathbb{C}$. Standard physics tricks tell us that it is no loss of generality to restrict ourselves to the center-of-mass subspace:

$$\mathbb{V}_{cm} = \{q \in \mathbb{C}^3 : \Sigma m_a q_a = 0\}.$$

On the center-of-mass subspace Lagrange showed that

$$(3) \quad I = \langle q, q \rangle := \Sigma m_a |q_a|^2$$

A standard computation, yields the Lagrange-Jacobi (or virial) identity

$$\ddot{I} = 4H$$

which is valid for any potential V which is homogeneous of degree -2 . It follows that on the energy level $H = 0$, the phase space function $\dot{I} = 2\langle q, v \rangle$ is an additional conserved quantity. If we do not want the size I of our solution to change then, we work on the invariant submanifold of phase space for which $H = 0, \dot{I} = 0$.

Our problem is to solve our Newton’s equations on the submanifold of phase space for which $H = 0, P = 0, \dot{I} = 0$ and $J = 0$.

2. REDUCTION AND SOLUTION.

The group G of rigid motions of the plane is a subgroup of the Galilean group and so maps solutions to solutions. As a consequence, the dynamics pushes down to the quotient space of \mathbb{C}^3 by G . This quotient space is called “shape space” and is homeomorphic to \mathbb{R}^3 . Points of shape space are oriented congruence classes of triangles. Shape space is endowed with standard “Hopf-Jacobi” coordinates w_1, w_2, w_3 which correspond to quadratic G -invariant functions on \mathbb{C}^3 . These coordinates have the property that

$$w_3 = \mu \Delta; \mu^2 = \frac{m_1 m_2 m_3}{m_1 + m_2 + m_3}$$

and $w_1^2 + w_2^2 + w_3^2 = \frac{I^2}{4}$. See [6] or [1]. Note that the equator $w_3 = 0$ corresponds to the degenerate triangles in which all three masses lie along a single line. We call $\pi : \mathbb{C}^3 \rightarrow \mathbb{R}^3; q \mapsto \pi(q) = w$ the “shape space projection”. (The formal process of pushing down the dynamics is “symplectic reduction” and requires fixing the value of the angular momentum J . We have already fixed the linear momentum P to be zero and will be fixing the angular momentum to be zero.)

Within shape space is the *shape sphere* S^2 , which is the sphere $|w|^2 = 1$ (or any sphere $I = I_0$ as long as I_0 is a positive constant). Points of this sphere are identified with oriented similarity classes of triangles, since scaling a triangle q by $\lambda > 0$ multiplies I by λ^2 . Alternatively, if we delete the origin from $\mathbb{V}_{cm} \cong \mathbb{C}^2$ and then quotient by complex scaling $q \mapsto \lambda q, \lambda \neq 0$ in \mathbb{C} , we arrive at $\mathbb{CP}^1 \cong S^2$. Now reflection about any line in the plane of the triangles has the effect $(w_1, w_2, w_3) \mapsto (w_1, w_2, -w_3)$ so that the upper hemisphere $w_3 \geq 0$ of the shape sphere $|w|^2 = 1$, realizes the space of similarity classes of triangles, with its boundary $w_3 = 0$ representing the degenerate collinear triangles.

The open upper hemisphere, $w_3 > 0$, $|w|^2 = \frac{I_0^2}{4}$, endowed with the metric $\frac{dw_1^2 + dw_2^2 + dw_3^2}{w_3^2}$ is a fairly well-known realization of the hyperbolic plane, sometimes referred to as the “Jemisphere model”. See pp. 69-71 of [2].

Theorem 1. *The shape space projection of those solutions to our Newton’s equations for which $H = \dot{I} = J = P = 0$ are geodesics for the Jemisphere model of the hyperbolic plane. Relative to the standard Hopf-Jacobi induced Cartesian coordinates w_1, w_2, w_3 on shape space described above, these projected solution curves are obtained by intersecting a hemisphere $w_1^2 + w_2^2 + w_3^2 = (1/4)I_0^2$, $w_3 > 0$ or $w_3 < 0$ with a vertical plane $Aw_1 + Bw_2 = \text{const.}$ (See figure 1.)*

Remark 1. The reflection $(w_1, w_2, w_3) \mapsto (w_1, w_2, -w_3)$ maps the “upper Jemisphere model” $w_3 > 0$ canonically isometrically to the lower Jemisphere model $w_3 < 0$, both having the same form above for the metric.

Remark 2. As a consequence of the expression of the projected solutions $w(t) = \pi(q(t))$, we see that when such a solution is extended over its maximum time range (a, b) of existence, it begins at one collinear configuration and ends at another. (In particular $\lim_{t \rightarrow a, b} w_3(t) = 0$.) One can compute the angle between the lines containing these two extremal collinear configurations by using the area formula, as per the periodicity proof for the figure eight solution given in [1].

3. PROOF

Step 1. Jacobi-Maupertuis. The Jacobi-Maupertuis principle, applied for energy $H = 0$, asserts that the zero-energy solutions to our Newton’s equations are, up to reparameterization, geodesics for the metric

$$ds_{JM}^2 = 2U(q)|dq|_E^2, \text{ where } U = -V$$

on \mathbb{C}^3 . The center-of-mass zero subspace is totally geodesic for this metric (conservation of linear momentum) and so we restrict the metric to this linear subspace $V \cong \mathbb{C}^2 \subset \mathbb{C}^3$.

Step 2. Symmetry reduction. Observe that ds_{JM}^2 is invariant under complex scaling: $q \mapsto \lambda q, \lambda \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. As a result, the metric ds_{JM}^2 descends to the quotient $\mathbb{C}^2 \setminus \{0\}/\mathbb{C}^*$ which is \mathbb{CP}^1 , the shape sphere. When we use this pushed-down metric the quotient projection map $\mathbb{V}_{cm} \setminus \{0\} \rightarrow \mathbb{CP}^1$ becomes a Riemannian submersion. We can identify this quotient projection with the shape projection π composed with the radial projection $w \mapsto \frac{1}{|w|}(w)$ from the shape space minus the origin onto a shape sphere. The Riemannian submersion property implies that geodesics for ds_{JM}^2 which are orthogonal to the \mathbb{C}^* -fibers project onto geodesics for the quotient metric. Now use the facts that $\dot{I} = 2\langle q, v \rangle$, $J = \langle v, iq \rangle$, and ds_{JM}^2 is conformal to the mass metric ds_E^2 , to conclude that a curve is orthogonal to the fiber if and only if $\dot{I} = 0 = J$ along that curve to see that our Newton solutions are projected to shape space geodesics for the pushed-down metric. It remains to compute this pushed down metric, let us call it $\pi_* ds_{JM}^2$, and show it is the Jemisphere metric. In [6], eq (43) it is shown that $ds_E^2 = \frac{|dw|^2}{I}$ when pushed down to the shape space. We have already seen that $\Delta = \mu w_3$ so that $U = \frac{\gamma \mu^2 I}{w_3^2}$. It follows that $\pi_* ds_{JM}^2 = \frac{\gamma \mu^2 |dw|^2}{w_3^2}$, which is the Jemisphere metric, as required. (The scaling constant $\gamma \mu^2$ changes the constant curvature to $-1/\mu\sqrt{\gamma}$ but does not change the geodesics.)

Finally we want to describe the geodesics for the Jemisphere metric in terms of our Hopf-Jacobi coordinates. Use the isometry between the Klein model and the Jemisphere metric which is projection along the w_3 -axis, as described in [2]. See figure 1. More specifically, the Klein model is realized as the disc $w_1^2 + w_2^2 < 1$ lying on the plane $w_3 = 1$ tangent to the sphere at the north pole. (We've assumed $I_0 = 4$ for simplicity so that $|w| = 1$.) Project points $(w_1, w_2, 1)$ of the Klein model to points (w_1, w_2, w_3) , of the hemisphere $w_1^2 + w_2^2 + w_3^2 = 1$, $w_3 > 0$ to get the isometry between models. (Note $w_3 = \sqrt{1 - w_1^2 - w_2^2}$.) Since geodesics of the Klein model are chords $Aw_1 + Bw_2 = C$ and since they map over to geodesics of the Jemisphere, we have that the same linear equations in w_1, w_2 characterize geodesics in the Jemisphere model.

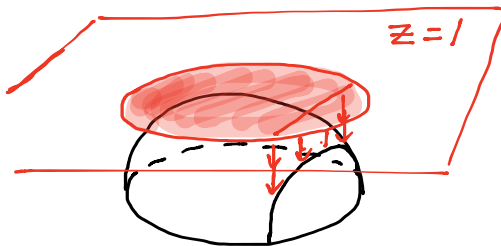


FIGURE 1. Projection along the vertical is an isometry between the Klein and Jemisphere models of the Hyperbolic Plane.

QED.

4. CONFESSION. MOTIVATION. HISTORY. OPEN PROBLEMS

I teach a geometry class nearly every year for an audience consisting primarily of future high school mathematics teachers. I feel an obligation to teach the rudiments of hyperbolic geometry. But almost all the students leave the class with no

understanding of what the hyperbolic plane is beyond a vague sense that “things get really tiny and squinched together when you approach the x-axis”. The present paper began as an attempt to provide a natural road in to hyperbolic geometry and its intuition for these students. I believe I have failed. Nevertheless I hope some readers find the exercise was interesting.

The papers [5] and [3] combined to suggest the approach taken here to building the “mechanical hyperbolic metric” of theorem 1. In [5] I applied the reductions of the present paper to the “strong-force” $1/r^2$ potential $V_2 = -\Sigma m_a m_b r_{ab}^2$ in the equal mass three-body case to obtain a metric on the shape sphere minus its three binary collision points, i.e. on the topologist’s pair-of-pants. The main theorem there is that this metric is complete with negative Gauss curvature everywhere except at two points (the equilateral triangles of Lagrange). As a corollary, the figure eight solution for that potential is unique up to isometry and scaling. In [3], Connor Jackman and I tried to extend the $N = 3$ hyperbolicity to the case $N = 4$ body problem with equal masses. The potential has the same form V_2 , except the sum is now taken over all 6 pairs amongst the 4 bodies. Connor proved that the results of [5] do not readily extend: the resulting 4-body JM metric has mixed curvature: some 2-planes have positive curvature, some negative.

The tricks we applied in [5], [3] and the present paper apply to any Galilean-invariant potential of homogeneity -2 on the N -body configuration space \mathbb{C}^N , provided that potential V is negative (possibly $-\infty$). The result is a metric on $\mathbb{CP}^{N-2} \setminus \Sigma$ where Σ is the set on which $V = -\infty$. That JM metric has the form $U ds_{FS}^2$ where $U = -V$.

Reflecting on the jump from $N = 3$ to $N=4$ in jumping from [5] to [3], it is natural to wonder what might happen when we take such a jump for our “3-point” potential (2). To define the analogous N -body potential we sum our 3-point potential over all triples of bodies. Thus, consider N point masses in the plane, labeling them $\{1, 2, \dots, N\}$. For each choice of 3 indices i, j, k out of $\{1, 2, \dots, N\}$, let $I(i, j, k)$ be the moment of inertia of the triangle formed by vertices q_i, q_j, q_k and let $\Delta(i, j, k)$ be the signed area of this triangle. Then the N -body analogue of our potential is

$$V_N = -\gamma \Sigma_{i,j,k} \frac{I(i, j, k)}{\Delta(i, j, k)^2}$$

the sum being over all three-element subsets $\{i, j, k\} \subset \{1, 2, \dots, N\}$. We observe that the singularity of V_N is precisely the set Σ_N of all “non-generic” planar N -gons, where we say that an N -gon is non-generic if some 3 of its vertices are collinear. Σ_N is a real codimension 1 subvariety which cuts the configuration space into a number of components.

Question 1. For $N = 4$ is the resulting JM metric on $\mathbb{CP}^2 \setminus \Sigma_4$ hyperbolic?

One computes without much difficulty that this JM metric is complete. Indeed, near any typical point of Σ_N we can choose a coordinate w_3 such that $w_3 = 0$ locally defines Σ and the JM metric as $w_3 \rightarrow 0$ has leading asymptotics that of the hyperbolic metric: $\frac{1}{w_3^2} ds_{Euc}^2$. There are restrictions on the topology of complete hyperbolic manifolds. If the manifold is simply connected then it must be diffeomorphic to the ball in that dimension. This suggests looking into the topology of $\mathbb{CP}^2 \setminus \Sigma_4$. We have verified that $\mathbb{CP}^2 \setminus \Sigma_4$ consists of 14 components, and that each one of which is diffeomorphic to B^4 , providing weak circumstantial evidence that the answer might be “yes” to question 1.

Jumping further ahead to higher N :

Question 2. How many components are there in the space $\mathbb{CP}^{N-2} \setminus \Sigma_N$ of general position planar N -gons? Is each component diffeomorphic to an open ball of dimension $2(N-2)$?

Question 3. Do potentials of the form of eq (2) arise in any physical or chemical problems of interest?

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